

Transfer group for renormalized multiple zeta values

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Abstract

We describe in this work all solutions to the problem of renormalizing multiple zeta values at arguments of any sign in a quasi-shuffle compatible way. As a corollary we clarify the relation between different renormalizations at non-positive values appearing in the recent literature.

Résumé

Groupe de transfert pour les valeurs zêta multiples renormalisées. Nous décrivons dans cette note l'ensemble des renormalisations possibles en des entiers de signe quelconque pour les valeurs zêta multiples, telles que les valeurs renormalisées vérifient les relations de quasi-battage. En corollaire nous clarifions la relation entre les différentes renormalisations aux entiers négatifs récemment apparues dans la littérature.

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Version française abrégée

Les *valeurs zêta multiples*, définies pour des entiers $k_1 \geq 2, k_2, \dots, k_n \geq 1$ par la somme itérée

$$\zeta(k_1, \dots, k_n) := \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}, \quad (1)$$

vérifient plusieurs familles de relations polynomiales, parmi lesquelles les relations de quasi-battage, qui se déduisent directement de la représentation (1), par exemple :

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a + b).$$

La définition (1) s'étend aux arguments complexes (s_1, \dots, s_n) . La *fonction zêta multiple* ainsi obtenue est holomorphe dans le domaine $\text{Re}(s_1 + \dots + s_j) > j$, $j = 1, \dots, n$. En profondeur $n \geq 1$, la fonction zêta multiple s'étend méromorphiquement à \mathbb{C}^n avec pôles donnés par :

$$\begin{aligned} s_1 &= 1, \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \dots, \\ s_1 + \dots + s_j &\in \mathbb{Z}_{\leq j} \text{ for } j \geq 3. \end{aligned}$$

Plusieurs extensions compatibles avec les relations de quasi-battage ont été proposées, pour des arguments dans tout \mathbb{Z} [12], ou seulement pour des arguments négatifs [6], ou strictement négatifs [4]. Nous montrons dans cette note que toutes les solutions au problème s'obtiennent à partir de l'une d'entre elles via l'action libre et transitive d'un groupe de transfert, obtenu comme groupe des caractères du quotient de l'algèbre de Hopf des quasi-battages (sur l'alphabet \mathbb{Z} , $\mathbb{Z}_{\leq 0}$ ou $\mathbb{Z}_{< 0}$ selon le cas) par l'idéal (de Hopf) engendré par les *mots non-singuliers*, c'est-à-dire ceux associés aux valeurs zêta multiples qui peuvent s'obtenir par prolongement analytique.

Nous montrons que le groupe de transfert est un groupe pro-unipotent dont l'algèbre de Lie est de dimension infinie. L'espace homogène principal des valeurs zêta multiples renormalisées s'identifie donc lui aussi à un espace vectoriel de dimension infinie. Le point essentiel réside dans la structure des mots non-singuliers, qui sont invariants par contraction de tout bloc de lettres consécutives, et qui engendrent linéairement un co-idéal N qui est de plus un co-idéal à gauche pour le coproduit de déconcaténation réduit.

1 Introduction

Multiple zeta values (MZVs), which are defined for $k_1 \geq 2, k_2, \dots, k_n \geq 1$ by the nested sum

$$\zeta(k_1, \dots, k_n) := \sum_{m_1 > \dots > m_n > 0} \frac{1}{m_1^{k_1} \dots m_n^{k_n}}, \quad (2)$$

have a rich and deep mathematical structure [16,9,1,15]. They appeared in a 1981 preprint of J. Écalle, in the context of resurgence theory in complex analysis [5]. A systematic study however started only a decade later with the works of D. Zagier [14] and M. E. Hoffman [7]. M. Kontsevich noted that (2) has a representation in terms of iterated integrals, which manifests itself in shuffle relations. We will focus on quasi-shuffle relations, directly derived from the nested sum representation (2), for example, $\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(b, a) + \zeta(a+b)$. The two representations combined yield intricate relations, which are commonly referred to as double shuffle structures underlying MZVs [10].

Definition (2) extends to complex arguments (s_1, \dots, s_n) . The *multiple zeta function* thus obtained is holomorphic in the domain $\text{Re}(s_1 + \dots + s_j) > j$, $j = 1, \dots, n$. In [11] it was shown that the MZV function can be meromorphically continued to \mathbb{C}^n with singularities

$$\begin{aligned} s_1 &= 1, \\ s_1 + s_2 &= 2, 1, 0, -2, -4, \dots \\ s_1 + \dots + s_j &\in \mathbb{Z}_{\leq j} \quad \text{for } j \geq 3. \end{aligned}$$

This raises the natural question of how to extend MZVs to any tuple of arguments in \mathbb{Z} , such that the quasi-shuffle relations are preserved. The solution to this so-called renormalization problem for MZVs is not unique: L. Guo and B. Zhang gave an answer for nonpositive arguments in [6]. Another approach through q -analogues of multiple zeta values, leading to a one-parameter family of values at negative arguments, has been proposed by the first three authors in [4]. An approach involving arguments in \mathbb{Z} (including mixed signs) has been developed by S. Paycha and the second author in [12]. The values obtained differ from each other. For example we have the following values for $\zeta(-1, -3)$:

$$\zeta_{GZ}(-1, -3) = \frac{83}{64512}, \quad \zeta_{EMS}(-1, -3) = \frac{121}{94080}, \quad \zeta_{MP}(-1, -3) = \frac{1}{840}.$$

Note that $\zeta_{EMS}(-1, -3)$ corresponds to the special value $t = 1$ of the family

$$\zeta_{EMS,t}(-1, -3) = \frac{1}{8064} \frac{166t^2 + 166t + 31}{(4t + 3)(4t + 1)},$$

which is defined for $t \in \{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$. The aforementioned solutions to the renormalization problem of MZVs all apply the Connes–Kreimer theory [3] on the quasi-shuffle Hopf algebra. The different values reflect different regularization methods.

In this note, we describe how all possible quasi-shuffle compatible renormalizations for arguments in \mathbb{Z} are related by a transfer group. More precisely, they form a set on which the transfer group acts freely and transitively. This group appears to be the character group of a commutative Hopf algebra, which is obtained as a quotient of a quasi-shuffle Hopf algebra by the (Hopf) ideal generated by *non-singular words*, i.e. words associated to multiple zeta values obtainable by analytic continuation. The same approach can be used for multiple zeta values restricted to nonpositive or negative arguments. All solutions above are thus on the same orbit under the action of the corresponding restricted transfer group.

2 Algebraic framework

We quickly describe the quasi-shuffle Hopf algebra [8] and the set of non-singular words. Let $Y := \{z_k : k \in \mathbb{Z}\}$ be the set of letters. Further let $\mathcal{H} := \langle Y \rangle_{\mathbb{Q}}$ be the free \mathbb{Q} -algebra generated by Y . The empty word is denoted by $\mathbf{1}$. We define the *quasi-shuffle product* $*$: $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H}$ by

- (i) $\mathbf{1} * w := w * \mathbf{1} := w$,
- (ii) $z_m u * z_n v := z_m(u * z_n v) + z_n(z_m u * v) + z_{m+n}(u * v)$,

for any $w, u, v \in \mathcal{H}$ and $m, n \in \mathbb{Z}$. The unit map is given by $u : \mathbb{Q} \rightarrow \mathcal{H}$, $u(\lambda) = \lambda \mathbf{1}$. The coproduct $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ is given by the deconcatenation

$$\Delta(w) := \sum_{uv=w} u \otimes v \quad (3)$$

for any word $w \in \mathcal{H}$ and the counit $\varepsilon : \mathcal{H} \rightarrow \mathbb{Q}$ is defined by $\varepsilon(\mathbf{1}) = 1$, and $\varepsilon(w) = 0$ for any $w \neq \mathbf{1}$. The reduced coproduct is given by $\tilde{\Delta}(w) := \Delta(w) - w \otimes \mathbf{1} - \mathbf{1} \otimes w$ for any $w \neq \mathbf{1}$. In Sweedler's notation we have $\tilde{\Delta}(w) = \sum_{(w)} w' \otimes w''$.

Since $(\mathcal{H}, *, \Delta)$ is a filtered and connected bialgebra it is automatically a Hopf algebra with antipode $S : \mathcal{H} \rightarrow \mathcal{H}$ given by $S(\mathbf{1}) = \mathbf{1}$ and

$$S(w) = -w - \sum_{(w)} S(w') * w'' = -w - \sum_{(w)} w' * S(w'') \quad (4)$$

for any word $w \in \ker(\varepsilon)$.

A *contraction of a block* in a word $w = z_{k_1} z_{k_2} z_{k_3} \cdots z_{k_n}$ refers to combining consecutive letters in w to a single letter, obtained by adding their indices together, which gives rise to a shorter word. For instance, two possible contractions in w are $w_1 = z_{k_1+k_2} z_{k_3} \cdots z_{k_n}$ or $w_2 = z_{k_1} z_{k_2+k_3+k_4} z_{k_5} \cdots z_{k_n}$.

A word $w = z_{k_1} \cdots z_{k_n}$ with letters from the alphabet Y will be called *non-singular* if the following conditions are verified:

- (i) $k_1 \neq 1$,
- (ii) $k_1 + k_2 \notin \{2, 1, 0, -2, -4, \dots\}$,
- (iii) $k_1 + \cdots + k_j \notin \mathbb{Z}_{\leq j}$ for $j \geq 3$.

Let $N \subset \mathcal{H}$ be the \mathbb{Q} -vector space spanned by the non-singular words. It is naturally graded by the length, $N = \bigoplus_{l \geq 1} N_l$. We will denote by N_n the space of \mathbb{Q} -linear combinations of non-singular words of length n . For any word $w = z_{k_1} \cdots z_{k_n} \in N$, the corresponding multiple zeta value

$$\zeta^*(w) := \zeta(k_1, \dots, k_n)$$

is either convergent or can be defined by analytic continuation [11].

Lemma 2.1 *The vector space N is a left coideal for the reduced deconcatenation coproduct $\tilde{\Delta}$. Moreover, N is invariant under contractions.*

PROOF. This is straightforward and left to the reader. \square

As a corollary, it follows that N is a two-sided coideal for the full deconcatenation coproduct Δ defined in (3).

3 Renormalized multiple zeta values and the transfer group

Let \mathcal{A} be a commutative unital algebra, and let $\mathcal{L}(\mathcal{H}, \mathcal{A})$ be the vector space of linear functions from \mathcal{H} to \mathcal{A} . The set $G_{\mathcal{A}}$ of unital algebra morphisms from \mathcal{H} to \mathcal{A} is a group with respect to the convolution product $\star: \mathcal{L}(\mathcal{H}, \mathcal{A}) \otimes \mathcal{L}(\mathcal{H}, \mathcal{A}) \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{A})$ defined by $\phi \otimes \psi \mapsto \phi \star \psi := m_{\mathcal{A}} \circ (\phi \otimes \psi) \circ \Delta$ and with unit $e := u \circ \varepsilon$. The inverse of $\phi \in G_{\mathcal{A}}$ is given by $\phi^{-1} = \phi \circ S$.

Lemma 3.1 *The set $T_{\mathcal{A}} := \{\phi \in G_{\mathcal{A}}: \phi|_N = 0\}$ is a subgroup of $(G_{\mathcal{A}}, \star, e)$.*

PROOF. Obviously, $e \in T_{\mathcal{A}}$. Let $\phi, \psi \in T_{\mathcal{A}}$. Since $G_{\mathcal{A}}$ is a group $\phi \star \psi^{-1} \in G_{\mathcal{A}}$. Further we observe for any $w \in N$ that

$$\begin{aligned} (\phi \star \psi^{-1})(w) &= (\phi \star (\psi \circ S))(w) = \phi(w) + \psi(S(w)) + \sum_{(w)} \phi(w')\psi(S(w'')) \\ &= \phi(w) - \psi(w) - \sum_{(w)} \psi(w')\psi(S(w'')) + \sum_{(w)} \phi(w')\psi(S(w'')) = 0 \end{aligned}$$

thanks to (4), $\psi \in G_{\mathcal{A}}$ and by means of Lemma 2.1. \square

The subgroup $T_{\mathcal{A}}$ will be called the *transfer group*. Now let $\zeta : N \rightarrow \mathcal{A}$ be a partially defined character. We define the set of all possible renormalizations with target algebra \mathcal{A} by

$$X_{\mathcal{A}, \zeta} := \{\alpha \in G_{\mathcal{A}} : \alpha|_N = \zeta\}.$$

Theorem 3.2 *The left group action $T_{\mathcal{A}} \times X_{\mathcal{A}, \zeta} \rightarrow X_{\mathcal{A}, \zeta}$ defined by $(\phi, \alpha) \mapsto \phi \star \alpha$ is free and transitive.*

PROOF. First we prove that the group action is well-defined. Let $\phi \in T_{\mathcal{A}}$ and $\alpha \in X_{\mathcal{A}, \zeta}$. For any $w \in N$ we obtain

$$(\phi \star \alpha)(w) = \phi(w) + \alpha(w) + \sum_{(w)} \phi(w')\alpha(w'') = \alpha(w) = \zeta(w)$$

using Lemma 2.1. The identity and compatibility relations of the group action are satisfied because $X_{\mathcal{A}, \zeta} \subset G_{\mathcal{A}}$.

Freeness is obvious. In order to prove transitivity, let $\alpha, \beta \in X_{\mathcal{A}, \zeta}$. Then for any $w \in N$, we have

$$\begin{aligned} (\alpha \star \beta^{-1})(w) &= (\alpha \star (\beta \circ S))(w) = \alpha(w) + \beta(S(w)) + \sum_{(w)} \alpha(w')\beta(S(w'')) \\ &= \alpha(w) - \beta(w) - \sum_{(w)} \beta(w')\beta(S(w'')) + \sum_{(w)} \alpha(w')\beta(S(w'')) \\ &= (\alpha - \beta)(w) + \sum_{(w)} (\alpha - \beta)(w')\beta(S(w'')) = 0, \end{aligned}$$

since $\beta \in G_{\mathcal{A}}$ and $\alpha|_N = \beta|_N = \zeta$. \square

It is easily seen that the two-sided ideal \mathcal{N} generated by N is a Hopf ideal of \mathcal{H} . The following result is straightforward.

Proposition 3.1 *For any commutative unital algebra \mathcal{A} , the transfer group $T_{\mathcal{A}}$ is isomorphic to the group of characters of the Hopf algebra \mathcal{H}/\mathcal{N} .*

Deconcatenation (3) is conilpotent, i.e. for any word w we have $\tilde{\Delta}^k(w) = 0$ for sufficiently large k , where $\tilde{\Delta}^k$ is the k -th iterated reduced coproduct. The same property holds for the reduced coproduct of \mathcal{H}/\mathcal{N} . Hence, by a theorem of D. Quillen [13] (see [2, Theorem 3.9.1]) the Hopf algebra \mathcal{H}/\mathcal{N} is isomorphic to the free commutative algebra on a vector space W , the image of \mathcal{H}/\mathcal{N} by the Eulerian idempotent $\pi_1 = \log^* \text{Id}$. As a result, the transfer group $T_{\mathcal{A}}$ is pro-nilpotent and can be identified with $\mathcal{L}(W, \mathcal{A})$ by multiplicative extension. Its Lie algebra $\mathfrak{t}_{\mathcal{A}}$ is the space of \mathcal{A} -valued infinitesimal characters of \mathcal{H}/\mathcal{N} , which identifies with $\mathcal{L}(W, \mathcal{A})$.

In the case ζ is given by the analytically continued multiple zeta values (with $\mathcal{A} = \mathbb{C}$), the set $X_{\mathbb{C}, \zeta}$ is not empty [12]. There are many more elements in $X_{\mathbb{C}, \zeta}$, due to the fact that the transfer group $T_{\mathbb{C}}$ is nontrivial:

Theorem 3.3 *The Lie algebra $\mathfrak{t}_{\mathcal{A}}$ of the transfer group is infinite-dimensional.*

PROOF. It will be convenient to use the shuffle product instead of the quasi-shuffle. Let \mathcal{H}_0 be the Hopf algebra $\langle Y \rangle_{\mathbb{Q}}$ endowed with the deconcatenation and the unital, associative, commutative shuffle product \sqcup satisfying

- (i) $\mathbf{1} \sqcup w := w \sqcup \mathbf{1} := w$,
- (ii) $z_m u \sqcup z_n v := z_m(u \sqcup z_n v) + z_n(z_m u \sqcup v)$.

It is the free commutative algebra over the free Lie algebra $\text{Lie}(Y)$ generated by the alphabet Y , and is isomorphic to \mathcal{H} by Hoffman's logarithm \log_H [8], which sends any word to a suitable linear combination of shorter words obtained by contractions. Our Hopf algebra \mathcal{H}/\mathcal{N} is isomorphic to $\mathcal{H}_0/\mathcal{N}_{(0)}$, where $\mathcal{N}_{(0)} = \log_H(\mathcal{N})$ is the ideal generated by $N_{(0)} = \log_H(N)$. Using the second assertion of Lemma 2.1 we easily obtain $N_{(0)} = N$, and then:

$$W = \text{Lie}(Y)/\pi_1(N),$$

where π_1 now stands for the Eulerian idempotent of \mathcal{H}_0 . The natural grading on \mathcal{H}_0 by depth generates a grading on W . The first component W_1 is one-dimensional, and there are obviously infinitely many linearly independent elements in $W_2 = \text{Lie}(Y)_2/\pi_1(N_2)$. A basis of W_2 can be identified with $\{[z_k, z_l], k > l \text{ and } z_k z_l \text{ is a singular word of length two}\}$. \square

Acknowledgements

The first author is supported by a Ramón y Cajal research grant from the Spanish government. DM, JS and JZ would like to thank the ICMAT for its warm hospitality and gratefully acknowledge support by the Severo Ochoa

Excellence Program. The second author is partially supported by Agence Nationale de la Recherche (projet CARMA).

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